Small-world effects in the majority-vote model

Paulo R. A. Campos* and Viviane M. de Oliveira†
Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560-970 São Carlos, SP, Brazil and Digital Life Laboratory, California Institute of Technology, Mail Code 136-93, Pasadena, California 91125

F. G. Brady Moreira‡
Departamento de Física, Universidade Federal de Pernambuco, 50670-901 Recife, PE, Brazil
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We investigate the majority-vote model on small-world networks by rewiring the two-dimensional square lattice. We observe that the introduction of long-range interactions does not remove the critical character of the model, that is, the system still exhibits a well-defined phase transition. However, we find that now the critical point is a monotonically increasing function of the rewiring probability. Moreover, we find that small-world effects change the class of universality of the model.

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I. INTRODUCTION

Regular lattices and random graphs [1] are appropriate to describe the topology of most systems studied in condensed matter physics. However, they do not represent well the topology of several other systems found in nature. The small-world (SW) networks try to capture the main features observed in some real networks, ranging from biological to social interacting systems [2,3]. The resulting networks interpolate between an ordered finite-dimensional lattice and completely random graphs. Like regular lattices, small-world networks can be highly clustered, but they also possess small characteristic path lengths, similar to those found in random networks [2,3].

Here, we address the influence of SW effects on the isotropic majority-vote model. The majority-vote model is a simple nonequilibrium spin system with up-down symmetry [4–7]. Besides its relevance for statistical mechanics, it is an interesting model for social behavior. When we apply spin models to social systems, a SW network is more appropriate as an underlying topology than a regular lattice [8]. In the majority-vote model, each individual is influenced by its neighborhood. Individuals have the tendency to accept the choice of the majority in their neighborhood with probability p_{agree}, and take the opposite view with probability $q=1-p_{\text{agree}}$. Previous investigations of the majority-vote model on regular lattices showed that it exhibits a phase transition with critical exponents that fall into the same class of universality as the equilibrium Ising model [5,7].

Our objective in this work is to identify the critical character of the majority-vote model when we introduce long-range interactions. In particular, we use Monte Carlo simulations and standard finite-size scaling techniques to determine the critical exponents γ/ν and β/ν, as well as the critical curve for several values of the rewiring probability.

The remainder of the paper is organized in the following way: In Sec. II, we describe the isotropic majority-vote model. In Sec. III, we define the relevant physical quantities used in our analysis. In Sec. IV, we present our results and discussions. And finally, in Sec. V, we present our conclusions.

II. MAJORITY-VOTE MODEL

In the isotropic majority-vote model, we ascribe a spin variable $\sigma_i=\pm 1$ to each site on the lattice. We update the system as follows: We choose sequentially a spin and determine the sign of the majority of the spins that are immediate neighbors of the chosen spin. With probability p_{agree}, the chosen spin is given this sign. With probability $q=1-p_{\text{agree}}$, the chosen spin is given the opposite sign. In the following, we also refer to q as the noise parameter. In terms of q, the probability of flipping is given by

$$w(\sigma_i)=\frac{1}{2}\left[1-(1-2q)\sigma_i S\left(\sum_{\delta=1}^{z}\sigma_{i+\delta}\right)\right], \tag{1}$$

where $S(x)=\text{sgn}(x)$ for $x\neq 0$, $S(0)=0$, δ denotes the nearest neighbor vector, and z is the coordination number. In a regular square lattice, the neighborhood of a site consists of its four nearest neighbors. The probability (1) exhibits "up-down" symmetry, i.e., $w(\sigma_i)=w(-\sigma_i)$ under the change of states of the Ising spins in the neighborhood of σ_i. For $q=0$, the model corresponds to the Ising model at temperature $T=0$.

In previous investigations, Monte Carlo simulations and mean-field calculations have shown that the majority-vote model presents a phase transition from an ordered to a disordered state at a critical value of the noise parameter, $q=q_c$, which depends on the lattice topology [5–7]. Furthermore, the corresponding critical phenomenon is in the same class of universality as the equilibrium Ising model [9], and so the critical exponents depend only on the lattice dimensionality. For the two-dimensional regular square lattice, q_c is ≈ 0.075 [5,7]. There are no previous results for the majority-vote model on random graphs.

*Electronic address: prac@caltech.edu
†Electronic address: viviane1@caltech.edu
‡Electronic address: brady@df.ufpe.br

In this paper, we consider the majority-vote model on a SW network. With probability p', we rewire each of the z bonds connecting site i to its nearest neighbors. In this way, we can test a bond twice, which means that the actual value of the rewiring probability p (as defined in the original model [2]), is related to p' by $p=p'^2+2p'(1-p')$. In fact, we carried out simulations using both the standard procedure and the one we employ here, and we found results that were completely equivalent. "Rewiring" in this context means that we move one end of the bond to a new randomly chosen site, i.e., we introduce some amount of long-range interactions (short cuts). Thus, the summation in Eq. (1) does not consider nearest neighbors only, as in the case of a regular lattice, but it runs over all those sites connected to spin i. Moreover, the rewiring algorithm alters the coordination number of the sites, although the mean connectivity is still equal to z.

III. RELEVANT PHYSICAL QUANTITIES

In order to delineate the critical behavior of the model, we analyze in the magnetization M_L, the magnetic susceptibility χ_L, and the Binder's fourth-order cumulant U_L, defined, respectively, by

$$M_L=\langle\langle m\rangle_S\rangle_C=\left\langle \left\langle \frac{1}{N}\left|\sum_1^N \sigma_i\right|\right\rangle_S\right\rangle_C, \qquad (2)$$

$$\chi_L=N[\langle\langle m^2\rangle_S\rangle_C-\langle\langle m\rangle_S\rangle_C^2], \qquad (3)$$

and

$$U_L=1-\frac{\langle\langle m^4\rangle_S\rangle_C}{3\langle\langle m^2\rangle_S\rangle_C^2}, \qquad (4)$$

where $N=L^2$ is the total number of sites, $\langle\cdots\rangle_S$ denotes that the averages are taken in the stationary regime, and $\langle\cdots\rangle_C$ means configurational averages.

The above quantities are functions of the noise parameter q and satisfy the following finite-size scaling relations [5]:

$$M_L(q)=L^{-\beta/\nu}\tilde{M}(L^{1/\nu}\varepsilon), \qquad (5)$$

$$\chi_L(q)=L^{\gamma/\nu}\tilde{\chi}(L^{1/\nu}\varepsilon), \qquad (6)$$

$$U_L(q)=\tilde{U}(L^{1/\nu}\varepsilon), \qquad (7)$$

where $\varepsilon=q-q_c$.

In Table I, we provide parameters of our Monte Carlo simulations. The difference between the total time and the equilibration time corresponds to the number of Monte Carlo steps (MCS) we use to estimate the averages for each run. The equilibration time is the number of MCS necessary to make the system reach the steady state regime. In our simulation, one Monte Carlo step is accomplished after updating all $N=L^2$ spins. We have checked that the results do not change significantly when we choose longer equilibration time or total time of simulation. The simulations were performed using the standard C random generator. For all sets of parameters, we have generated ten distinct configurations of SW networks, and we have simulated ten independent Monte Carlo runs for each distinct configuration.

TABLE I. System size, and the corresponding total time of simulation and equilibration time.

Lattice size	Total time	Equilibration time
$L=30$	3000 MCS	2000 MCS
$L=50$	5000 MCS	4000 MCS
$L=70$	7000 MCS	5000 MCS
$L=100$	10000 MCS	7000 MCS

IV. RESULTS

In Fig. 1, we plot the magnetization M_L as a function of the noise parameter q. In part (a) we show M_L for distinct system sizes L and for fixed $p=0$, which corresponds to the isotropic majority-vote model on a regular square lattice as discussed in Ref. [5]. We clearly notice that there is a phase transition from an ordered state $(M_L>0)$ to a disordered state $(M_L\approx 0)$. As expected for critical systems, the magnetization displays a sharper transition when we consider larger values of L. For $q>q_c$, the magnetization M_L vanishes in the limit $L\to\infty$, whereas for $q<q_c$ it has a finite value. The critical exponents fall into the same class of universality as the equilibrium Ising model [5–7].

In Fig. 1(b), we demonstrate the effect of long-range interactions in the network. From the plot, we can see that with increased rewiring probability p, the order parameter M_L shows a smoother transition as compared to the behavior

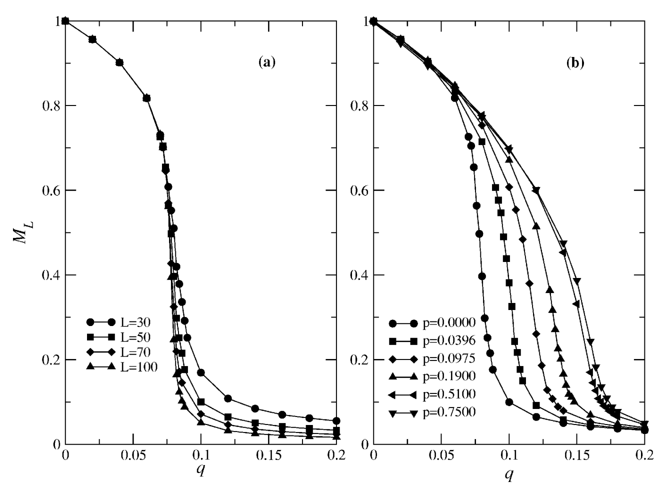

FIG. 1. Magnetization as a function of q. In part (a), we have $p=0$ (regular lattice) and the different curves correspond to different lattice sizes. From top to bottom: $L=30$, $L=50$, $L=70$, and $L=100$. In part (b), we have a fixed system size of $L=50$, and the lines correspond to distinct values of the rewiring probability. From left to right we have $p=0$, 0.04, 0.1, 0.19, 0.51, and 0.75.

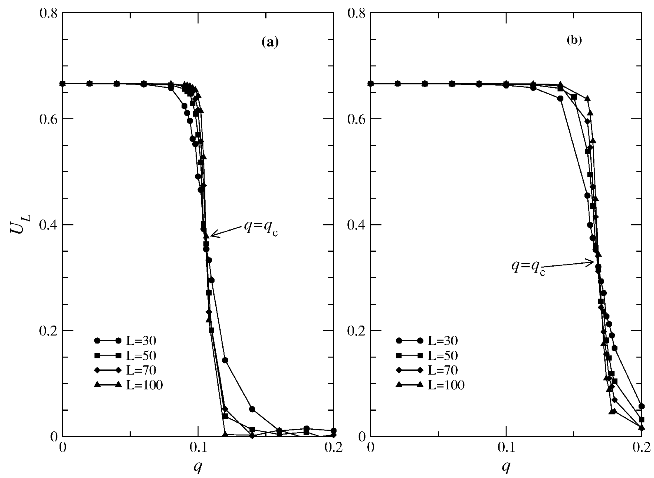

FIG. 2. Binder's fourth-order cumulant as a function of q. In part (a), $p=0.04$ and (b), $p=0.75$.

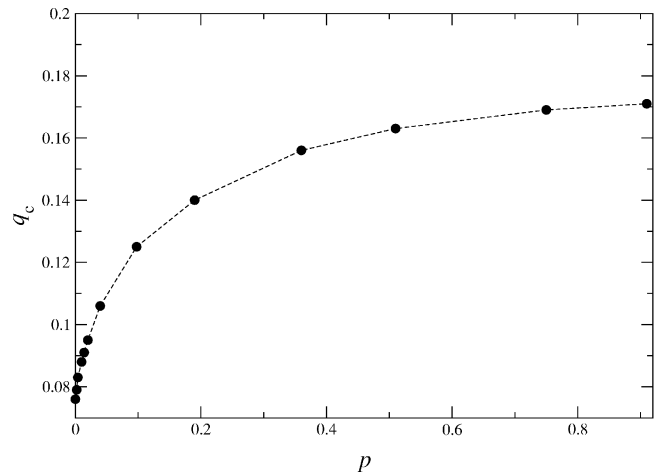

FIG. 3. Phase diagram. The critical values of the noise parameter q_c as a function of the concentration of long-range interactions p for the majority-vote model on two-dimensional SW networks. The line is a guide to the eye.

seen for $p=0$. We also observe that considerable deviation occurs even for a small concentration of long-range interactions. We notice a collapse of the curves in the regime of a high concentration of long-range bonds (high p), that is, the data points for $p>0.75$ fall over the curve for $p=0.75$. Moreover, the presence of long-range interactions in the model results in a larger robustness of the system against noise. In a social context, this means that the propagation of information and the influence of the neighborhood is more effective with long-range interactions. This effect can be easily understood in terms of our daily experience. We are most easily convinced if several people who belong to distinct groups agree on a particular opinion. Similar high efficiency of information exchange has been observed in a variety of systems with small-world topology, such as neural systems, communication networks, and transport networks [10].

In Fig. 2, we plot Binder's fourth-order cumulant U_L. From the analysis of U_L, we determine the critical point $q_c(p)$ of the model. The critical point $q_c(p)$ is estimated as the point at which the different curves for different system sizes L intercept each other (see figure). We show the typical behavior of U_L for two distinct values of the probability p. In part (a), we have $p=0.04$ and in part (b), we have $p=0.75$.

In Fig. 3, we show the phase diagram q_c versus p. Our measurements of q_c have an error of 0.002, which corresponds to a percentage error in the range of 1.2–2.5%. We observe that the increase of q_c is more pronounced for small values of p. Recent studies on the Ising model on SW networks found similar qualitative dependence of the critical temperature on the parameter p [11,12].

In order to study the universality of the model, we also investigated its critical exponents. In Fig. 4, we display the value of the maximum of the susceptibility, $\chi_{\max,L}$, versus L in a log-log scale for several values of p. From the slopes of the straight lines, which correspond to the best fits to the data points, we estimated the corresponding values of the critical ratio γ/ν. We find that the introduction of long-range interactions changes the values of the critical exponents of the model. For $p=0$ (the regular lattice), we obtained $\gamma/\nu=1.69$, which is in a good agreement with previous results [5]. However, for $p>0$, the critical exponents change along the critical line $q=q_c(p)$. In fact, we obtained a variation of γ/ν from its value at $p=0$ up to the value $\gamma/\nu=1.11\pm0.06$ for $p>0.01$. For complementarity, we also calculated the ratio β/ν. The ratio β/ν is obtained through the L dependence of the magnetization calculated at the critical value $q_c(p)$, as indicated by Eq. (5). Our analysis yielded p-dependent values for the critical ratio β/ν satisfying (within error bars of about 5%) the hyperscaling relation, i.e., $2\beta/\nu=d-\gamma/\nu$, where $d=2$ is the dimensionality of the lattice. So, the present results indicate that the majority-vote models defined on a regular square lattice and on small-world networks are in different universality classes.

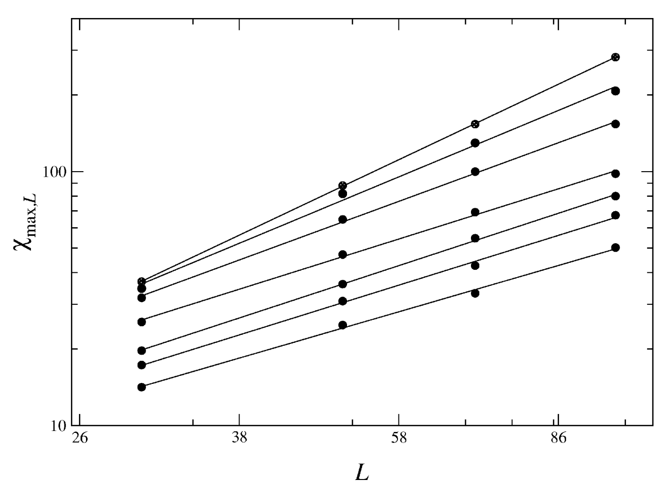

FIG. 4. Maximum value of the susceptibility χ_L versus L in a log-log scale. The different data correspond to distinct values of p. From top to bottom, $p=0.0$, 0.004, 0.01, 0.04, 0.1, 0.19, and 0.51. The exponent values γ/ν are 1.69(1), 1.49(7), 1.31(3), 1.12(4), 1.17(2), 1.11(4), and 1.04(4), respectively.

V. CONCLUSION

We have investigated the phase diagram and critical behavior of the majority-vote model on small-world networks through Monte Carlo simulations and finite-size scaling analysis. The phase diagram indicates that the presence of long-range interactions in the system results in a larger robustness of the system against noise. For the critical behavior, we found critical exponents that are dependent on the fraction of short cuts introduced in the system. In contrast to the studies of the ferromagnetic transition [11–14] on small-world networks, which suggest mean-field-like behavior for the transition, here we have obtained a well-defined phase transition, indicating that the majority-vote models defined on a regular square lattice and on small-world networks belong to different universality classes.

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